STABILITY OF A QUARTER AUTOMOTIVE WITH NONLINEAR QUADRATIC SUSPENSION

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ABSTRACT – In our paper we discuss the stability of the equilibrium position for a quarter automotive with nonlinear quadratic suspension. For the stable positions of equilibrium we also obtain the approximations of the small oscillations. The theory is applied to a realistic numerical example.

1. INTRODUCTION

Usually, the suspension of a quarter of automobile is represented by two masses linked one to another by a linear spring, one mass being connected to the ground by another nonlinear spring. The two masses represent the wheel and the corresponding part of the entire automobile [1, 2, 3, 4]. There are only a few approaches, which deal with nonlinear suspensions [5, 6, 7]. Some papers deal with time delay feedback [8] or magnetorheological dampers [9]. In our paper we consider that the wheel can be simulated by a mass and a nonlinear quadratic spring.

2. MODEL OF SUSPENSION

The model considered in this paper is captured in Fig. 1. The elastic force in the spring 1 is considered to be given by

$$F = k_1 z + \varepsilon_1 z^2, \tag{1}$$

where z is its elongation.

Isolating the two masses, one obtains the differential equations of motion

$$m_1 \ddot{x}_1 = k_2(x_2 - x_1) - k_1 x_1 - \varepsilon_1 x_1^2 - m_1 g,\ m_2 \ddot{x}_2 = -k_2(x_2 - x_1) - m_2 g. \tag{2}$$

With the notations

$$x_1 = \xi_1,\ x_2 = \xi_2,\ \dot{x}_1 = \xi_3,\ \dot{x}_2 = \xi_4, \tag{3}$$

the system (2) is brought to a system of four nonlinear differential equations of first order

$$\dot{\xi}_1 = \xi_3,\ \dot{\xi}_2 = \xi_4,\ \dot{\xi}_3 = \frac{1}{m_1}\left[k_2(\xi_2 - \xi_1) - k_1\xi_1 - \varepsilon_1\xi_1^2 - m_1 g\right],$$
$$\dot{\xi}_4 = -\frac{1}{m_2}\left[k_2(\xi_2 - \xi_1) + m_2 g\right]. \tag{4}$$

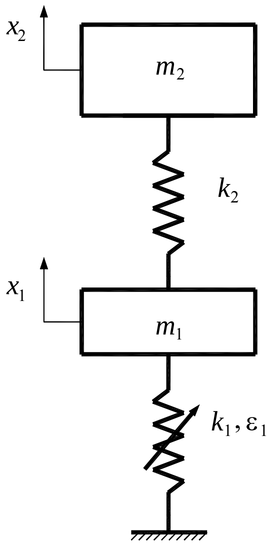

Fig. 1. A quarter–car model.

3. EQUILIBRIUM POSITIONS

The equilibrium positions are obtained at the intersections of the nullclines; hence, the following system is deduced

$$\xi_3 = 0, \; \xi_4 = 0, \; k_2(\xi_2 - \xi_1) - k_1\xi_1 - \varepsilon_1\xi_1^2 - m_1 g = 0, \; k_2(\xi_2 - \xi_1) + m_2 g = 0. \quad (5)$$

Subtracting the last two equations, one gets

$$\varepsilon_1\xi_1^2 + k_1\xi_1 + (m_1 + m_2)g = 0. \quad (6)$$

Making now $\xi_1 \mapsto -\xi_1$, one obtains

$$\varepsilon_1\xi_1^2 - k_1\xi_1 + (m_1 + m_2)g = 0. \quad (7)$$

Analyzing the last two equations and applying the Descartes theorem, the following statements hold true:

– if $\varepsilon_1 > 0$, then the equation (6) has zero or two negative roots;

– if $\varepsilon_1 < 0$, then the equation (6) has one positive root and one negative root.

Denoting by Δ the discriminate of the equation (6),

$$\Delta = k_1^2 - 4\varepsilon_1(m_1 + m_2)g, \quad (8)$$

We can say:

– if $0 < \varepsilon_1 < \dfrac{k_1^2}{4(m_1 + m_2)g}$, then the system has two equilibrium position given by

$$\bar{\xi}_1 = \frac{-k_1 - \sqrt{\Delta}}{2\varepsilon_1} < 0\,,\; \bar{\bar{\xi}} = \frac{-k_1 + \sqrt{\Delta}}{2\varepsilon_1} < 0,\; \bar{\xi}_1 < \bar{\bar{\xi}}_1\,; \tag{9}$$

– if $\varepsilon_1 = \dfrac{k_1^2}{4(m_1 + m_2)g}$, then the system has only one equilibrium position (the equation (6) has a double negative root) given by

$$\bar{\xi}_1 = \bar{\bar{\xi}}_1 = -\frac{k_1}{2\varepsilon_1}\,; \tag{10}$$

– if $\varepsilon_1 < 0$, then the system has two equilibrium position given by

$$\bar{\xi}_1 = \frac{-k_1 + \sqrt{\Delta}}{2\varepsilon_1} < 0\,,\; \bar{\bar{\xi}} = \frac{-k_1 + \sqrt{\Delta}}{2\varepsilon_1} > 0\,,\; \left|\bar{\xi}_1\right| < \bar{\bar{\xi}}_1\,; \tag{11}$$

– if $\varepsilon_1 = 0$, then the equation (6) becomes a linear one with the solution

$$\bar{\xi}_1 = -\frac{(m_1 + m_2)g}{k_1} < 0\,. \tag{12}$$

From the last expression (5) we get the value ξ_2 at the equilibrium.

4. STABILITY OF THE EQUILIBRIA

The Jacobi matrix for the system (4) reads

$$[\mathbf{J}] = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ j_{31} & j_{32} & 0 & 0 \\ j_{41} & j_{42} & 0 & 0 \end{bmatrix}, \tag{13}$$

where

$$j_{31} = -\frac{k_2}{m_1} - \frac{k_1}{m_1} - \frac{2\varepsilon_1\xi_1}{m_1},\; j_{32} = \frac{k_2}{m_1},\; j_{41} = \frac{k_2}{m_2},\; j_{42} = -\frac{k_2}{m_2}\,. \tag{14}$$

The characteristic equation

$$\det(\mathbf{J} - \lambda\mathbf{I}) = 0 \tag{15}$$

where $\mathbf{I}$ is the four order unity matrix, takes the form

$$\begin{vmatrix} -\lambda & 0 & 0 & 0 \\ 0 & -\lambda & 0 & 1 \\ j_{31} & j_{32} & -\lambda & 0 \\ j_{41} & j_{42} & 0 & -\lambda \end{vmatrix} = 0 \tag{16}$$

where from

$$\begin{vmatrix} j_{31} - \lambda^2 & j_{32} \\ j_{41} & j_{42} - \lambda^2 \end{vmatrix} = 0, \tag{17}$$

that is

$$\lambda^4 - (j_{31} + j_{42})\lambda^2 + j_{31}j_{42} - j_{41}j_{32} = 0, \tag{18}$$

which is a bi–square algebraic equation.

The discriminate of this equation is

$$\Delta = (j_{31} + j_{42})^2 - 4(j_{31}j_{42} - j_{41}j_{32}) = (j_{31} - j_{42})^2 + 4j_{41}j_{32}. \tag{19}$$

The equation (18) has all the roots pure, distinct, imaginary if and only if

$$\Delta > 0,\ j_{31} + j_{42} < 0,\ j_{31}j_{42} - j_{41}j_{32} > 0. \tag{20}$$

Keeping into account the relations (14), we get

$$j_{31} - j_{42} = -\frac{k_2}{m_1} - \frac{k_1}{m_1} - \frac{2\varepsilon_1\xi_1}{m_1} + \frac{k_2}{m_2}, \tag{21}$$

$$j_{31} + j_{42} = -\frac{k_2}{m_1} - \frac{k_1}{m_1} - \frac{2\varepsilon_1\xi_1}{m_1} - \frac{k_2}{m_2}, \tag{22}$$

$$j_{41}j_{32} = \frac{k_2^2}{m_1m_2}, \tag{23}$$

$$j_{31}j_{42} - j_{41}j_{32} = \frac{k_1k_2}{m_1m_2} + \frac{2\varepsilon_1k_2\xi_1}{m_1m_2}, \tag{24}$$

hence, the conditions (20) become

$$\left(-\frac{k_2}{m_1} - \frac{k_1}{m_1} - \frac{2\varepsilon_1\xi_1}{m_1} + \frac{k_2}{m_2}\right)^2 + 4\frac{k_2^2}{m_1m_2} > 0, \tag{25}$$

$$\frac{k_2}{m_1} + \frac{k_1}{m_1} + \frac{2\varepsilon_1\xi_1}{m_1} + \frac{k_2}{m_2} > 0, \tag{26}$$

$$k_1 + 2\varepsilon_1\xi_1 > 0\,, \tag{27}$$

where ξ_1 is given by one of the expressions (9), (10), (11) or (12).

If $\varepsilon_1 = 0$, then the conditions (25), (26) and (27) are fulfilled and it results that in the linear case the equilibrium is simply stable.

Let us observe that the condition (25) is always true.

If $\varepsilon_1 = \dfrac{k_1^2}{4(m_1 + m_2)g}$ and ξ_1 is given by the expression (10), then the conditions (26) and (27)lead to

$$\frac{k_2}{m_1} + \frac{k_1}{m_1} > 0\,, \tag{28}$$

$$0 > 0\,, \tag{29}$$

and, because the relation (29) is false, it results that the equilibrium is unstable.

If $\varepsilon_1 < 0$, then, considering the root $\overline{\xi}_1 < 0$ (given by (11)), we observe that the conditions (26) and (27) hold true and it results that this equilibrium position is simply stable. For the root $\overline{\overline{\xi}}_1$ from (11), the condition (27) leads to

$$-\sqrt{\Delta} > 0\,, \tag{30}$$

which is false, i.e. the corresponding equilibrium position is unstable.

If $0 < \varepsilon_1 < \dfrac{k_1^2}{4(m_1 + m_2)g}$ and ξ_1 is given by $\overline{\xi}_1$ in (9), then the condition (27) leads to the same relation (30), hence the equilibrium is unstable. For $\overline{\overline{\xi}}_1$ the condition (27) is true and the condition (26) offers

$$\frac{k_2}{m_1} + \frac{k_1}{m_1} + \frac{\sqrt{\Delta}}{m_1} > 0\,, \tag{31}$$

an obviously true relation, hence the equilibrium is simply stable

5. SMALL OSCILLATIONS AROUND THE STABLE EQUILIBRIUM POSITIONS

Let us return to the system (4) and let us give sufficiently small in norm perturbations to the parameters ξ_i, i.e. $\xi_i \mapsto \xi_i + \zeta_i$, $i = \overline{1,4}$. One thus obtains the system in deviations

$$\dot{\zeta}_1 = \zeta_3\,,\ \dot{\zeta}_2 = \zeta_4\,,\ \dot{\zeta}_3 = \frac{1}{m_1}\left[k_2(\zeta_2 - \zeta_1) - k_1\zeta_1 + 2\xi_1\zeta_1 + \xi_1^2\right],\ \dot{\zeta}_4 = -\frac{k_2}{m_2}(\zeta_2 - \zeta_1) \tag{32}$$

or, by linearization,

$$\dot{\zeta}_1 = \zeta_3\,,\ \dot{\zeta}_2 = \zeta_4\,,\ \dot{\zeta}_3 = \frac{1}{m_1}\left[-(k_2 + k_1 - 2\xi_1)\zeta_1 + k_2\zeta_2\right],\ \dot{\zeta}_4 = \frac{k_2}{m_2}\zeta_1 - \frac{k_2}{m_2}\zeta_2\,. \tag{33}$$

Denoting now

$$a_{11} = \frac{k_2 + k_1 - 2\xi_1}{m_1}\,,\ a_{12} = -\frac{k_2}{m_1}\,,\ a_{21} = -\frac{k_2}{m_2}\,,\ a_{22} = \frac{k_2}{m_2}\,, \tag{34}$$

we get the system

$$\ddot{\zeta}_1 + a_{11}\zeta_1 + a_{12}\zeta_2 = 0\,,\ \ddot{\zeta}_2 + a_{21}\zeta_1 + a_{22}\zeta_2 = 0\,. \tag{35}$$

From the first relation (35) one has

$$\zeta_2 = -\frac{1}{a_{12}}\left(\ddot{\zeta}_1 + a_{11}\zeta_1\right),\ \ddot{\zeta}_2 = -\frac{1}{a_{12}}\left(\zeta_1^{(iv)} + a_{11}\ddot{\zeta}_1\right). \tag{36}$$

und, replacing in the second relation (35), one obtains

$$\zeta_1^{(iv)} + (a_{11} + a_{22})\ddot{\zeta}_1 + (a_{11} + a_{22} - a_{12}a_{21})\zeta_1 = 0\,. \tag{37}$$

Denoting now

$$\Delta = (a_{11} - a_{22})^2 + 4a_{12}a_{21} > 0 \tag{38}$$

the eigenpulsations are given by

$$p_1 = \sqrt{\left|\frac{(a_{11} + a_{22}) + \sqrt{\Delta}}{2}\right|}\,,\ p_2 = \sqrt{\left|\frac{(a_{11} + a_{22}) - \sqrt{\Delta}}{2}\right|}\,. \tag{39}$$

6. NUMERICAL APPLICATION

Let us consider the values $m_1 = 50\,\text{kg}$, $m_2 = 250\,\text{kg}$, $k_1 = 160000\,\text{N/m}$, $k_2 = 16000\,\text{N/m}$, $\varepsilon_1 = 2 \cdot 10^6\,\text{N/m}^2$, $g = 9.8065\,\text{m/s}^2$.

Let us observe that

$$0 < \varepsilon_1 \frac{k_1^2}{4(m_1 + m_2)g} = 2.1754 \cdot 10^6\,. \tag{40}$$

and the equilibrium positions are

$$\overline{\xi}_1 = -0.0514\,\mathrm{m}\,,\ \overline{\overline{\xi}}_1 = -0.0286\,\mathrm{m} \tag{41}$$

first being unstable and the second simply stable.

Moreover, the parameters given by (34) are (for $\overline{\overline{\xi}}_1$)

$$a_{11} = 3.52\cdot 10^3\,,\ a_{12} = -320\,,\ a_{21} = -64\,,\ a_{22} = 64\,, \tag{42}$$

Δ has the value

$$\Delta = 12025856 \tag{43}$$

and the eigenpulsations are

$$p_1 = 59.379\,\mathrm{s}^{-1}\,,\ p_2 = 7.622\,\mathrm{s}^{-1}\,. \tag{44}$$

The numerical simulation was performed for the initial data $\xi_1^{(0)} = -0.03\,\mathrm{m}$, $\zeta_2^{(0)} = -0.19\,\mathrm{m}$, $\xi_3^{(0)} = 0\ \mathrm{m/s}$, $\xi_4^{(0)} = 0\ \mathrm{m/s}$ and it is captured in Fig. 2.

The reader can observe the good agreement between the numerical results and the theory.

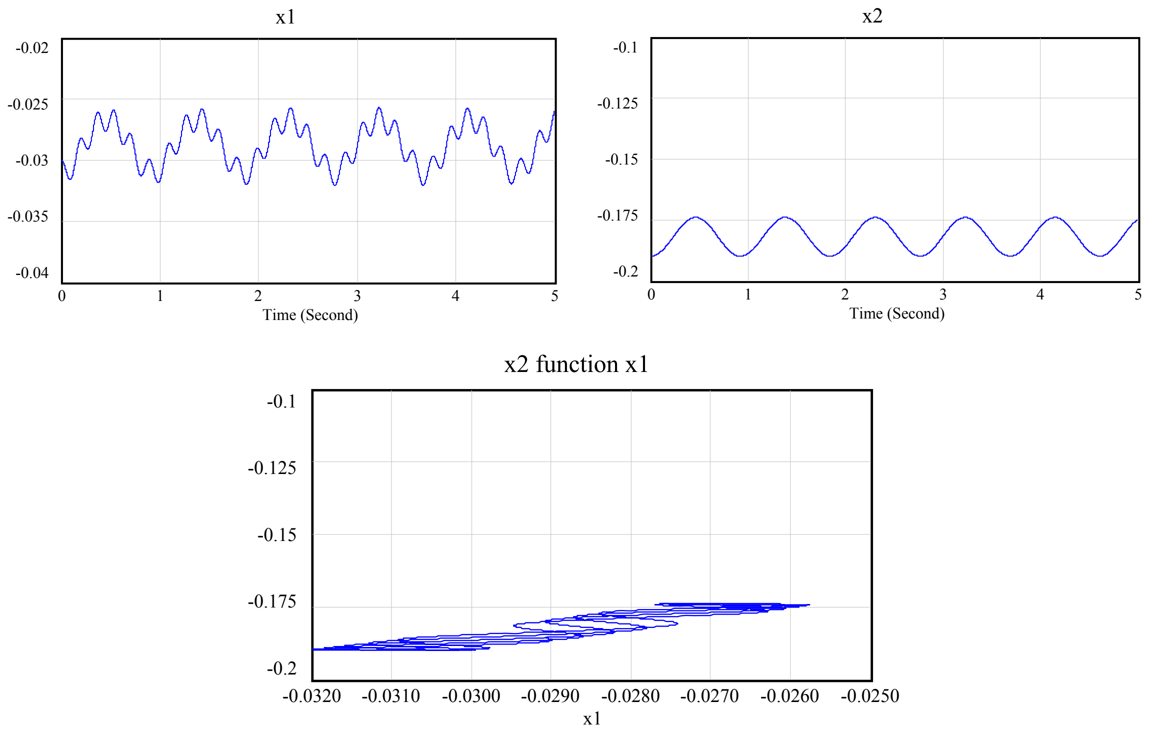

Fig.2. The results of the numerical simulation; $x_1 = x_1(t)$ for $0 \le t \le 5\,\mathrm{s}$, $x_2 = x_2(t)$ for $0 \le t \le 5\,\mathrm{s}$, $x_2 = x_2(x_1)$ $0 \le t \le 2\,\mathrm{s}$

7. CONCLUSIONS

In our paper we presented a nonlinear model for a quarter–car. We treated all the equilibrium positions and we discussed their stability in the most general case. The theory is confirmed by the numerical simulations.

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