BEHAVIOR OF A QUARTER-CAR SUSPENSION WITH DAMPER AND QUADRATIC NONLINEARITY

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ABSTRACT – In this paper we obtain the analytical solution for a quarter-car suspension with damper and quadratic nonlinear spring. The method used here is the multiple scales of time method.

1. INTRODUCTION

The problem of the nonlinear suspension for the automobiles is an open one. The difficulties appear from the nonlinear character of the differential equations of motion, which leads to numerical solution. The analytical form is obtained by different methods, all these methods leading to development into series.

2. MATHEMATICAL MODEL

The model of the suspension is drawn in Fig. 1, where the force in the spring 1 is given by

$$F = k_1\lambda + \mu_1\lambda^2, \tag{1}$$

λ being the elongation of the spring.

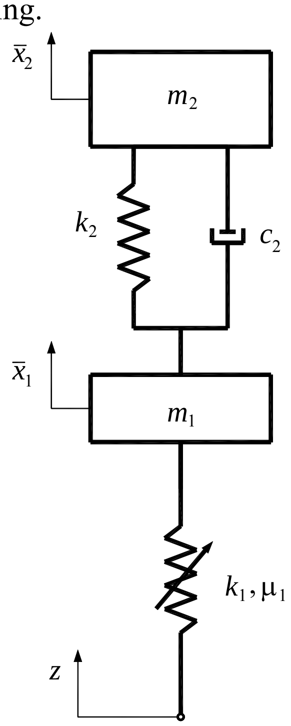

Fig. 1. Mathematical model.

The equations of motion read

$$m_1\ddot{\bar{x}}_1 + k_2(\bar{x}_2 - \bar{x}_1) + c_2(\dot{\bar{x}}_2 - \dot{\bar{x}}_1) - k_1(\bar{x}_1 - z) - \mu_1(\bar{x}_1 - z)^2 - m_1 g \,,$$
$$m_2\ddot{\bar{x}}_2 = -k_2(\bar{x}_2 - \bar{x}_1) - c_2(\dot{\bar{x}}_2 - \dot{\bar{x}}_1) - m_2 g \,. \tag{2}$$

In these equations, z marks the excitation from the ground.

Making the change of variables $\bar{x}_1 \mapsto x_1 + z$, $\bar{x}_2 \mapsto x_2$, one obtains the system

$$m_1\ddot{x}_1 + m_1\ddot{z} = k_2(x_2 - x_1 - z) + c_2(\dot{x}_2 - \dot{x}_1 - \dot{z}) - k_1 x_1 - \mu_1 x_1^2 - m_1 g \,,$$
$$m_2\ddot{x}_2 = -k_2(x_2 - x_1 - z) - c_2(\dot{x}_2 - \dot{x}_1 - \dot{z}) - m_2 g \tag{3}$$

or, equivalent,

$$\ddot{x}_1 + \frac{k_1 + k_2}{m_1}x_1 - \frac{k_2}{m_1}x_2 + \frac{c_2}{m_1}\dot{x}_1 - \frac{c_2}{m_1}\dot{x}_2 = -\frac{k_2}{m_1}z - \frac{c_2}{m_1}\dot{z} - \frac{\mu_1}{m_1}x_1^2 - g \,,$$
$$\ddot{x}_2 - \frac{k_2}{m_2}x_1 + \frac{k_2}{m_2}x_2 - \frac{c_2}{m_2}\dot{x}_1 + \frac{c_2}{m_2}\dot{x}_2 = \frac{k_2}{m_2}z + \frac{c_2}{m_2}\dot{z} - g \,. \tag{4}$$

3. NORMAL FORM

Let us denote

$$a_{11} = \frac{k_1 + k_2}{m_1}\,,\; a_{12} = -\frac{k_2}{m_1}\,,\; a_{21} = -\frac{k_2}{m_2}\,,\; a_{22} = \frac{k_2}{m_2} \tag{5}$$

and let $\mathbf{A}$ be the matrix

$$\mathbf{A} = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}. \tag{6}$$

Its characteristic equation reads

$$\begin{vmatrix} a_{11} - \lambda & a_{12} \\ a_{21} & a_{22} - \lambda \end{vmatrix} = 0\,, \tag{7}$$

therefore

$$\lambda^2 - (a_{11} + a_{22})\lambda + a_{11}a_{22} - a_{12}a_{21} = 0\,. \tag{8}$$

We have

$$a_{11} + a_{22} = \frac{k_1 + k_2}{m_1} + \frac{k_2}{m_2}\,, \tag{9}$$

$$a_{11}a_{22} - a_{12}a_{21} = \frac{k_1 k_2}{m_1 m_2} \tag{10}$$

and the equation (8) takes the form

$$\lambda^2 - \left(\frac{k_1 + k_2}{m_1} + \frac{k_2}{m_2} \right) \lambda + \frac{k_1 k_2}{m_1 m_2} = 0 , \tag{11}$$

for which the discriminant reads

$$\Delta = \left(\frac{k_1 + k_2}{m_1} + \frac{k_2}{m_2} \right)^2 - \frac{4 k_1 k_2}{m_1 m_2} . \tag{12}$$

If $\Delta > 0$, then the eigenvalues are

$$\lambda_{1,2} = \frac{\left(\frac{k_1 + k_2}{m_1} + \frac{k_2}{m_2} \right) \pm \sqrt{\Delta}}{2} > 0 . \tag{13}$$

In this case there exists a matrix $\mathbf{P}$ formed with the eigenvectors of $\mathbf{A}$, so that

$$\mathbf{P}^{-1}\mathbf{A}\mathbf{P} = \begin{bmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{bmatrix} . \tag{14}$$

With the transformation

$$\mathbf{x} = \mathbf{P}\mathbf{y} \tag{15}$$

the system (4) takes the normal form

$$\begin{gathered} \ddot{y}_1 + \omega_1^2 y_1 + \alpha_1 \dot{y}_1 - \alpha_1 \dot{y}_2 = \gamma_{11} y_1^2 + \gamma_{12} y_1 y_2 + \gamma_{22} y^2 + \delta_1 z + \delta_2 \dot{z} + g_1 , \\ \ddot{y}_2 + \omega_2^2 y_2 - \beta_1 \dot{y}_1 + \beta_1 \dot{y}_2 = \varepsilon_1 z + \varepsilon_2 \dot{z} + g_2 . \end{gathered} \tag{16}$$

4. METHOD OF MULTIPLE SCALES OF TIME

We suppose that

$$y_1 = \varepsilon u_1(T_0, T_1, T_2, \ldots) + \varepsilon^2 u_2(T_0, T_1, T_2, \ldots) + \varepsilon^3 u_3(T_0, T_1, T_2, \ldots) + \ldots , \tag{17}$$

$$y_2 = \varepsilon v_1(T_0, T_1, T_2, \ldots) + \varepsilon^2 v_2(T_0, T_1, T_2, \ldots) + \varepsilon^3 v_3(T_0, T_1, T_2, \ldots) + \ldots , \tag{18}$$

where

$$T_i = \varepsilon^i t , \; i = 0, 1, 2, \ldots \tag{19}$$

Moreover, the excitation from the ground is assumed to be

$$z = z_0 \cos(\omega t) \tag{20}$$

We can write

$$\frac{\mathrm{d}y_1}{\mathrm{d}t} = \varepsilon \frac{\partial u_1}{\partial T_0} + \varepsilon^2 \left(\frac{\partial u_1}{\partial T_1} + \frac{\partial u_2}{\partial T_0} \right) + \varepsilon^3 \left(\frac{\partial u_1}{\partial T_2} + \frac{\partial u_2}{\partial T_1} + \frac{\partial u_3}{\partial T_0} \right) + \ldots, \tag{21}$$

$$\frac{\mathrm{d}^2 y_1}{\mathrm{d}t^2} = \varepsilon \frac{\partial^2 u_1}{\partial T_0^2} + \varepsilon^2 \left(2\frac{\partial^2 u_1}{\partial T_0 \partial T_1} + \frac{\partial^2 u_2}{\partial T_0^2} \right) + \varepsilon^3 \left(2\frac{\partial^2 u_1}{\partial T_0 \partial T_2} + \frac{\partial^2 u_1}{\partial T_1^2} + 2\frac{\partial^2 u_2}{\partial T_0 \partial T_1} + \frac{\partial^2 u_3}{\partial T_0^2} \right) + \ldots \tag{22}$$

Similar formulas may be written for y_2.

Further on, we consider

$$\delta_1 = \varepsilon \delta_1^*,\ \delta_2 = \varepsilon \delta_2^*,\ g_1 = \varepsilon g_1^*,\ g_2 = \varepsilon g_2^*. \tag{23}$$

The equations (16) take the form

$$\begin{aligned}
&\varepsilon \frac{\partial^2 u_1}{\partial T_0^2} + \varepsilon^2 \left(2\frac{\partial^2 u_1}{\partial T_0 \partial T_1} + \frac{\partial^2 u_2}{\partial T_0^2} \right) + \varepsilon^3 \left(2\frac{\partial^2 u_1}{\partial T_0 \partial T_2} + \frac{\partial^2 u_1}{\partial T_1^2} + 2\frac{\partial^2 u_2}{\partial T_0 \partial T_1} + \frac{\partial^2 u_3}{\partial T_0^2} \right) + \\
&+ \omega_1^2 \left(\varepsilon u_1 + \varepsilon^2 u_2 + \varepsilon^3 u_3 \right) + \alpha_1 \left[\varepsilon \frac{\partial u_1}{\partial T_0} + \varepsilon^2 \left(\frac{\partial u_1}{\partial T_1} + \frac{\partial u_2}{\partial T_0} \right) + \varepsilon^3 \left(\frac{\partial u_1}{\partial T_2} + \frac{\partial u_2}{\partial T_1} + \frac{\partial u_3}{\partial T_0} \right) \right] - \\
&- \alpha_1 \left[\varepsilon \frac{\partial v_1}{\partial T_0} + \varepsilon^2 \left(\frac{\partial v_1}{\partial T_1} + \frac{\partial v_2}{\partial T_0} \right) + \varepsilon^3 \left(\frac{\partial v_1}{\partial T_2} + \frac{\partial v_2}{\partial T_1} + \frac{\partial v_3}{\partial T_0} \right) \right] = \gamma_{11} \left(\varepsilon u_1 + \varepsilon^2 u_2 + \varepsilon^3 u_3 \right)^2 + \\
&+ \gamma_{12} \left(\varepsilon u_1 + \varepsilon^2 u_2 + \varepsilon^3 u_3 \right)\left(\varepsilon v_1 + \varepsilon^2 v_2 + \varepsilon^3 v_3 \right) + \gamma_{22} \left(\varepsilon v_1 + \varepsilon^2 v_2 + \varepsilon^3 v_3 \right)^2 + \\
&+ \varepsilon \delta_1^* z_0 \cos(\omega t) - \varepsilon \delta_2^* z_0 \omega \sin(\omega t) + \varepsilon g_1^* + \ldots,
\end{aligned} \tag{24}$$

$$\begin{aligned}
&\varepsilon \frac{\partial^2 v_1}{\partial T_0^2} + \varepsilon^2 \left(2\frac{\partial^2 v_1}{\partial T_0 \partial T_1} + \frac{\partial^2 v_2}{\partial T_0^2} \right) + \varepsilon^3 \left(2\frac{\partial^2 v_1}{\partial T_0 \partial T_2} + \frac{\partial^2 v_1}{\partial T_1^2} + 2\frac{\partial^2 v_2}{\partial T_0 \partial T_1} + \frac{\partial^2 v_3}{\partial T_0^2} \right) + \\
&+ \omega_2^2 \left(\varepsilon v_1 + \varepsilon^2 v_2 + \varepsilon^3 v_3 \right) - \beta_1 \left[\varepsilon \frac{\partial u_1}{\partial T_0} + \varepsilon^2 \left(\frac{\partial u_1}{\partial T_1} + \frac{\partial u_2}{\partial T_0} \right) + \varepsilon^3 \left(\frac{\partial u_1}{\partial T_2} + \frac{\partial u_2}{\partial T_1} + \frac{\partial u_3}{\partial T_0} \right) \right] + \\
&+ \beta_1 \left[\varepsilon \frac{\partial v_1}{\partial T_0} + \varepsilon^2 \left(\frac{\partial v_1}{\partial T_1} + \frac{\partial v_2}{\partial T_0} \right) + \varepsilon^3 \left(\frac{\partial v_1}{\partial T_2} + \frac{\partial v_2}{\partial T_1} + \frac{\partial v_3}{\partial T_0} \right) \right] = \\
&= \varepsilon \varepsilon_1^* z_0 \cos(\omega t) - \varepsilon \varepsilon_2^* z_0 \omega \sin(\omega t) + \varepsilon g_2^* + \ldots
\end{aligned} \tag{25}$$

Separating for corresponding powers of ε, one gets
- for order ε:

$$\frac{\partial^2 u_1}{\partial T_0^2} + \omega_1^2 u_1 + \alpha_1 \frac{\partial u_1}{\partial T_0} - \alpha_1 \frac{\partial v_1}{\partial T_0} = \delta_1^* z_0 \cos(\omega t) - \delta_2^* z_0 \omega \sin(\omega t) + g_1^* ,$$
$$\frac{\partial^2 v_1}{\partial T_0^2} + \omega_2^2 v_1 - \beta_1 \frac{\partial u_1}{\partial T_0} + \beta_1 \frac{\partial v_1}{\partial T_0} = \varepsilon_1^* z_0 \cos(\omega t) - \varepsilon_2^* z_0 \omega \sin(\omega t) + g_2^* ; \tag{26}$$

- for order ε^2:

$$2\frac{\partial^2 u_1}{\partial T_0 \partial T_1} + \frac{\partial^2 u_2}{\partial T_0^2} + \omega_1^2 u_2 + \alpha_1\left(\frac{\partial u_1}{\partial T_1} + \frac{\partial u_2}{\partial T_0}\right) - \alpha_1\left(\frac{\partial v_1}{\partial T_1} + \frac{\partial v_2}{\partial T_0}\right) = \gamma_{11} u_1^2 + \gamma_{12} u_1 v_1 + \gamma_{22} v_1^2 ,$$
$$2\frac{\partial^2 v_1}{\partial T_0 \partial T_1} + \frac{\partial^2 v_2}{\partial T_0^2} + \omega_2^2 v_2 - \beta_1\left(\frac{\partial u_1}{\partial T_1} + \frac{\partial u_2}{\partial T_0}\right) + \beta_1\left(\frac{\partial v_1}{\partial T_1} + \frac{\partial v_2}{\partial T_0}\right) = 0 ; \tag{27}$$

-for order ε^3:

$$2\frac{\partial^2 u_1}{\partial T_0 \partial T_2} + \frac{\partial^2 u_1}{\partial T_1^2} + 2\frac{\partial^2 u_2}{\partial T_0 \partial T_1} + \frac{\partial^2 u_3}{\partial T_0^2} + \omega_1^2 u_3 + \alpha_1\left(\frac{\partial u_1}{\partial T_2} + \frac{\partial u_2}{\partial T_1} + \frac{\partial u_3}{\partial T_0}\right) -$$
$$-\alpha_1\left(\frac{\partial v_1}{\partial T_2} + \frac{\partial v_2}{\partial T_1} + \frac{\partial v_3}{\partial T_0}\right) = 2\gamma_{11} u_1 u_2 + \gamma_{12}(u_1 v_2 + u_2 v_1) + 2\gamma_{22} v_1 v_2 ,$$
$$2\frac{\partial^2 v_1}{\partial T_0 \partial T_2} + \frac{\partial^2 v_1}{\partial T_1^2} + 2\frac{\partial^2 v_2}{\partial T_0 \partial T_1} + \frac{\partial^2 v_3}{\partial T_0^2} + \omega_2^2 v_3 - \beta_1\left(\frac{\partial u_1}{\partial T_2} + \frac{\partial u_2}{\partial T_1} + \frac{\partial u_3}{\partial T_0}\right) +$$
$$+\beta_1\left(\frac{\partial v_1}{\partial T_2} + \frac{\partial v_2}{\partial T_1} + \frac{\partial v_3}{\partial T_0}\right) = 0 . \tag{28}$$

Particular solutions for the system (26) are

$$u_{1p} = C_0 + C_1 \cos(\omega t) + C_2 \sin(\omega t),\ v_{1p} = D_0 + D_1 \cos(\omega t) + D_2 \sin(\omega t), \tag{29}$$

where C_i and D_i, $i = \overline{0, 2}$, are real constants.

Considering now the homogenous system (26)

$$\frac{\partial^2 u_1}{\partial T_0^2} + \omega_1^2 u_1 + \alpha_1 \frac{\partial u_1}{\partial T_0} - \alpha_1 \frac{\partial v_1}{\partial T_0} = 0 ,\ \frac{\partial^2 v_1}{\partial T_0^2} + \omega_2^2 v_1 - \beta_1 \frac{\partial u_1}{\partial T_0} + \beta_1 \frac{\partial v_1}{\partial T_0} = 0 \tag{30}$$

and looking for the solutions in the forms

$$u_1 = A_1 e^{r_1 T_0} ,\ v_1 = B_1 e^{r_2 T_0} , \tag{31}$$

where r_1 and r_2 are constant values, while A_1 and B_1 do not depend on T_0, one gets

$$r_1^2 A_1 e^{r_1 T_0} + \omega_1^2 A_1 e^{r_1 T_0} + \alpha_1 A_1 e^{r_1 T_0} - \alpha_1 B_1 e^{r_2 T_0} = 0 ,$$
$$r_2^2 B_1 e^{r_2 T_0} + \omega_2^2 B_1 e^{r_2 T_0} - \beta_1 A_1 e^{r_1 T_0} + \beta_1 B_1 e^{r_2 T_0} = 0 , \tag{32}$$

wherefrom

$$\frac{r_1^2 A_1 + \omega_1^2 A_1}{r_2^2 B_1 + \omega_2^2 B_1} \mathrm{e}^{(r_1 - r_2)T_0} = -\frac{\alpha_1}{\beta_1} = \text{constant}\,; \tag{33}$$

hence $r_1 = r_2 = r$ and the system (32) becomes

$$A_1\left(r^2 + \omega_1^2 + \alpha_1 r\right) - B_1\alpha_1 = 0\,,\ B_1\left(r^2 + \omega_2^2 - \beta_1 r\right) + B_1\beta_1 = 0\,. \tag{34}$$

This algebraic system has nonzero solutions if and only if its determinant is equal to zero

$$\begin{vmatrix} r^2 + \omega_1^2 + \alpha_1 r & -\alpha_1 \\ r^2 + \omega_2^2 - \beta_1 r & \beta_1 \end{vmatrix} = 0\,, \tag{35}$$

i.e.

$$\left(\alpha_1 + \beta_1\right) r^2 + \beta_1\omega_1^2 + \alpha_1\omega_2^2 = 0\,, \tag{36}$$

wherefrom

$$r_{1,2} = \pm\sqrt{-\frac{\alpha_1\omega_2^2 + \beta_1\omega_1^2}{\alpha_1 + \beta_1}}\,. \tag{37}$$

It results

$$u_1 = A_{11}\left(T_1, T_2, \ldots\right)\mathrm{e}^{r_1 T_0} + A_{12}\left(T_1, T_2, \ldots\right)\mathrm{e}^{r_2 T_0}\,,\ v_1 = B_{11}\left(T_1, T_2, \ldots\right)\mathrm{e}^{r_1 T_0} + B_{12}\left(T_1, T_2, \ldots\right)\mathrm{e}^{r_2 T_0} \tag{38}$$

and the general solution reads

$$\begin{aligned} u_1 &= A_{11}\,\mathrm{e}^{r_1 T_0} + A_{12}\,\mathrm{e}^{r_2 T_0} + C_0 + C_1\cos(\omega t) + C_2\sin(\omega t)\,, \\ v_1 &= B_{11}\,\mathrm{e}^{r_1 T_0} + B_{12}\,\mathrm{e}^{r_2 T_0} + D_0 + D_1\cos(\omega t) + D_2\sin(\omega t)\,. \end{aligned} \tag{39}$$

In the steady state motion the values r_1 and r_2 are complex and the solution (39) can be put in the form

$$\begin{aligned} u_1 &= A_1\,\mathrm{e}^{\mathrm{i}|r|T_0} + \overline{A}_1\,\mathrm{e}^{-\mathrm{i}|r|T_0} + C_0 + C_1\,\mathrm{e}^{\mathrm{i}\omega T_0} + \overline{C}_1\,\mathrm{e}^{-\mathrm{i}\omega T_0}\,, \\ v_1 &= B_1\,\mathrm{e}^{\mathrm{i}|r|T_0} + \overline{B}_1\,\mathrm{e}^{-\mathrm{i}|r|T_0} + D_0 + D_1\,\mathrm{e}^{\mathrm{i}\omega T_0} + \overline{D}_1\,\mathrm{e}^{-\mathrm{i}\omega T_0}\,, \end{aligned} \tag{40}$$

where A_1, B_1, C_1 and D_1 are complex parameters, the superior bar marks the complex conjugate, while A_1 and B_1 are functions of T_1, T_2, …

The procedure continues with the systems (27), (28).

5. CONCLUSIONS

This paper presented a method to obtain the approximation of the analytical solution for the nonlinear vibrations of a quarter-car. The idea was to use the normal form theory and the multiple scales of time method.

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