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THE METHOD OF EQUIVALENT LINEARIZATION FOR SYSTEMS SUBJECTED TO NON-STATIONARY RANDOM EXCITATION

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ABSTRACT - The method of equivalent linearization is applied to the general problem of the response of non-linear discrete systems to non-stationary random excitation. Conditions for minimum equation difference are determined which do not depend explicitly on time but only on the instantaneous statistics of the response process. Using the equivalent linear parameters, a deterministic non-linear ordinary differential equation for the covariance function is derived. The theoretical analyses are verified by numerical results. An example is given of a damped Duffing oscillator subjected to modulated white noise.

1. SYSTEM MODEL

To illustrate the procedure of equivalent linearization theory, let us consider the following oscillator with a nonlinear restoring force component. The ordinary differential equation of the motion can be written as:

$$m\ddot{\eta}(t)+c\dot{\eta}(t)+k\eta(t)+\alpha k\eta^3(t)=F(t) \qquad (1)$$

where m is the mass, c is the viscous damping coefficient, $F(t)$ is the external excitation signal with zero mean and $\eta(t)$ is the displacement response of the system.

The reduced equation is

$$\ddot{\eta}(t)+2\xi p\dot{\eta}(t)+p^2\eta(t)+p^2\alpha\eta^3(t)=f(t) \qquad (2)$$

where ξ is the critical damping factor, and p is the undamped natural frequency, for the linear system.

As a next let us consider excitation described by subsequent correlation function

$$R_F(\tau)=De^{-\lambda\tau}\cos\beta\tau\ , \qquad (3)$$

where parameters D>0, λ>0, $\beta\geq 0$.

Power spectral density function [1] of excitation we obtain from the relation:

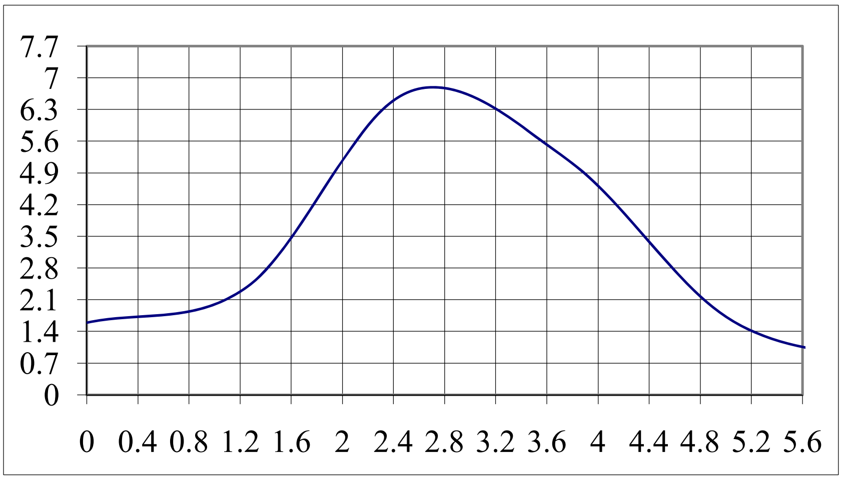
$$S_F(\omega)=\frac{1}{2\pi}\int_{-\infty}^{\infty}R_F(\tau)d\tau \qquad (4)$$

By substitution of the (3) in the (4) and integration we obtain

$$S_F(\omega)=\frac{D\lambda}{\pi}\frac{\omega^2+\lambda^2+\beta^2}{\left|(i\omega)^2+2\lambda(i\omega)+\lambda^2+\beta^2\right|^2} \qquad (5)$$

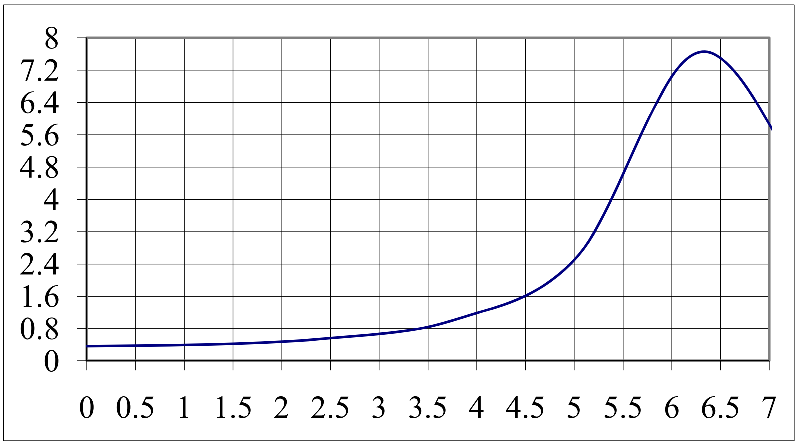
or

$$S_F(\omega)=\frac{D\lambda}{\pi}\frac{\omega^2+\lambda^2+\beta^2}{\left(\lambda^2+\beta^2-\omega^2\right)^2+4\lambda^2\omega^2}\,. \tag{6}$$



$$\frac{\omega}{2\pi}$$

Fig. 1. The power spectral density $S_F\ [N^2\cdot s]$ of excitation for $D=50\ N^2, \lambda=1s^{-1}, \beta=3s^{-1}$.



$$\frac{\omega}{2\pi}$$

Fig. 2. The power spectral density $S_F\ [N^2\cdot s]$ of excitation for $D=50N^2, \lambda=1s^{-1}, \beta=6{,}5s^{-1}$.

Power spectral density function of output we can obtain from the relation

$$S_\eta(\omega)=\frac{S_F(\omega)/m^2}{(p_e{}^2-\omega^2)^2+4\xi^2p^2\omega^2}\,. \tag{7}$$

So we obtain

$$S_\eta(\omega)=\frac{D\lambda(\omega^2+\lambda^2+\beta^2)}{\pi m^2\left\{\left[p^2-\omega^2+3\alpha p^2\sigma^2_\eta\right]^2+4\xi^2p^2\omega^2\right\}\left[\left(\lambda^2+\beta^2-\omega^2\right)^2+4\lambda^2\omega^2\right]}. \tag{8}$$

The displacement variance [2] of the single-degree of freedom system under Gaussian white noise excitation can be expressed as,

$$\sigma^2_{\eta} = R_{\eta}(0) = \int_{-\infty}^{\infty} S_{\eta}(\omega) d\omega . \qquad (9)$$

Substitution of the (8) in the (9) and obtain

$$\sigma^2_{\eta} = \frac{D\lambda}{m\pi} \int_{-\infty}^{\infty} \frac{(\omega^2+\lambda^2+\beta^2)}{\left\{\left[p^2-\omega^2+3\alpha p^2\sigma^2_{\eta}\right]^2+4\xi^2 p^2\omega^2\right\}\left[\left(\lambda^2+\beta^2-\omega^2\right)^2+4\lambda^2\omega^2\right]} d\omega. \qquad (10)$$

Integration [3,4] obtain

$$\int_{-\infty}^{\infty} \frac{\omega^2+d}{\left|(i\omega)^2+2\lambda(i\omega)+d\right|^2 \left|(i\omega)^2+b_1(i\omega)+b_0\right|^2} d\omega = \frac{\pi(b_o h_1+h_1 h_2-h_3)}{b_0(h_1 h_2 h_3-b_o h_1^2 d-h_3^3)}, \qquad (11)$$

where

$$h_1 = b_1+2\lambda, \quad h_2 = b_0+2\lambda b_1+d, \quad h_3 = 2\lambda b_0+db_1 . \qquad (12)$$

In this case

$$\begin{gathered} h_1 = 2(\xi p+\lambda), h_2 = p_e^2+4\lambda\xi p+\lambda^2+\beta^2, h_3 = 2\lambda p_e^2+2\xi p(\lambda^2+\beta^2) \\ b_0 = p_e^2 = p^2(1+3\alpha\sigma^2_{\eta}), b_1 = 2\xi p. \end{gathered} \qquad (13)$$

$$\sigma^2_{\eta} = \frac{A}{B}, \qquad (14)$$

where

$$\begin{gathered} A = D\lambda\sigma^2_{\eta}[12\alpha p^2(\xi p+\lambda)-6\alpha\lambda p^2]+D\lambda\{2p^2(\xi p+\lambda)+ \\ +8\lambda\xi p(\xi p+\lambda)+2(\xi p+\lambda)(\lambda^2+\beta^2)-2\lambda p^2-2\xi p(\lambda^2+\beta^2)\} \end{gathered} \qquad (15)$$

$$\begin{gathered} B = 108m\sigma^6_{\eta}\lambda\xi\alpha^3 p^7+36\sigma^4_{\eta} p^4 m\alpha\{\lambda\alpha[p^2+4\lambda\xi p+(\lambda^2+\beta^2)](\xi p+\lambda)+ \\ +2p(\xi p+\lambda)^2[p^2+4\lambda\xi p+(\lambda^2+\beta^2)][\lambda p+\xi(\lambda^2+\beta^2)]+\alpha[2p^2\lambda^2+ \\ +p\xi\lambda(\lambda^2+\beta^2)+(\xi p+\lambda)^2(\lambda^2+\beta^2)]+\lambda\alpha p^3\}+12m\sigma^2_{\eta}\left\{\{p^3\alpha(\xi p+\lambda)[\lambda p+\right. \\ +\xi(\lambda^2+\beta^2)][p^2+4\lambda\xi p+(\lambda^2+\beta^2)]-\xi^2 p^4\alpha(\lambda^2+\beta^2)-\xi p^5\lambda\alpha(\lambda^2+ \\ +\beta^2)-\lambda^2 p^6\alpha-p^4\alpha(\xi p+\lambda)^2(\lambda^2+\beta^2)\}+p^4\{\lambda\alpha[p^2+4\lambda\xi p+(\lambda^2+\beta^2)](\xi p+\lambda)+ \\ +2p(\xi p+\lambda)^2[p^2+4\lambda\xi p+(\lambda^2+\beta^2)][\lambda p+\xi(\lambda^2+\beta^2)]+\alpha[2p^2\lambda^2+ \\ \left.+p\xi\lambda(\lambda^2+\beta^2)+(\xi p+\lambda)^2(\lambda^2+\beta^2)]\}\right\}+4p^3 m(\xi p+\lambda)[\lambda p+\xi(\lambda^2+\beta^2)][p^2+ \\ +4\lambda\xi p+(\lambda^2+\beta^2)]-4\xi^2 p^4 m(\lambda^2+\beta^2)-4\xi p^5 m\lambda(\lambda^2+\beta^2)-4\lambda^2 mp^6- \\ -4p^4 m(\xi p+\lambda)^2(\lambda^2+\beta^2). \end{gathered} \qquad (16)$$

Using the notation

$$l = 108m\lambda\xi\alpha^3 p^7 \qquad (17)$$

$$\begin{gathered} n = 36p^4\alpha m\{\lambda\alpha[p^2+4\lambda\xi p+(\lambda^2+\beta^2)](\xi p+\lambda)+2p(\xi p+\lambda)^2[p^2+4\lambda\xi p+(\lambda^2+ \\ +\beta^2)][\lambda p+\xi(\lambda^2+\beta^2)]+\alpha[2p^2\lambda^2+p\xi\lambda(\lambda^2+\beta^2)+(\xi p+\lambda)^2(\lambda^2+\beta^2)]+\lambda\alpha p^3\} \end{gathered} \qquad (18)$$

$$r=12m\Big\{\{p^3\alpha(\xi p+\lambda)[\lambda p+\xi(\lambda^2+\beta^2)][p^2+4\lambda\xi p+(\lambda^2+\beta^2)]-\xi^2p^4\alpha(\lambda^2+\beta^2)- -\xi p^5\lambda\alpha(\lambda^2+\beta^2)-\lambda^2p^6\alpha-p^4\alpha(\xi p+\lambda)^2(\lambda^2+\beta^2)\}+p^4\{\lambda\alpha[p^2+4\lambda\xi p+(\lambda^2+ +\beta^2)](\xi p+\lambda)+2p(\xi p+\lambda)^2[p^2+4\lambda\xi p+(\lambda^2+\beta^2)][\lambda p+\xi(\lambda^2+\beta^2)]+\alpha[2p^2\lambda^2+ +p\xi\lambda(\lambda^2+\beta^2)+(\xi p+\lambda)^2(\lambda^2+\beta^2)]]\}\Big\} \quad (19)$$

$$s=4p^3m\Big\{(\xi p+\lambda)[\lambda p+\xi(\lambda^2+\beta^2)][p^2+4\lambda\xi p+(\lambda^2+\beta^2)]-4\xi^2p^4(\lambda^2+\beta^2)- -4\xi p^5\lambda(\lambda^2+\beta^2)-4\lambda^2p^6-4p^4(\xi p+\lambda)^2(\lambda^2+\beta^2)\Big\}-D\lambda[12\alpha p^2(\xi p+\lambda)-6\alpha\lambda p^2] \quad (20)$$

$$q=-D\lambda\{2p^2(\xi p+\lambda)-8\lambda\xi p(\xi p+\lambda)-2(\xi p+\lambda)(\lambda^2+\beta^2)+2\lambda p^2+2\xi p(\lambda^2+\beta^2)\} \quad (21)$$

obtain the equation

$$l\sigma^8_\eta+n\sigma^6_\eta+r\sigma^4_\eta+s\sigma^2_\eta+q=0\,. \quad (22)$$

We can always find a way to decompose the nonlinear restoring force to one linear component plus a nonlinear component

$$h(\eta)=p^2(\eta+G(\eta)\alpha)\,, \quad (23)$$

where α is the nonlinear factor to control the type and degree of nonlinearity in the system. The idea of linearization is replacing the equation by a linear system:

$$\ddot{\eta}(t)+2\xi_e p_e\dot{\eta}(t)+p_e^2\eta(t)=f(t), \quad (24)$$

where

$$\xi_e=\frac{p}{p_e}\xi. \quad (25)$$

is the damping ratio of equivalent linearized system and p_e is the natural frequency of the equivalent linearized system.

To find an expression for p_e, it is necessary to minimize the expected value of the difference between equations (2) and (24) in a least square sense. Now the difference is the difference between the nonlinear stiffness and linear stiffness terms , which is

$$e=h(\eta(t))-p^2_e\eta(t)\,. \quad (26)$$

The value of p_e can be obtained by minimizing the expectation , of the square error:

$$\frac{dE\{e^2\}}{dp_e^{\,2}}=0\,. \quad (27)$$

Substituting the equation (26) into equation (27) performing the necessary differentiation, the expression of p_e can be obtained as:

$$p_e^{\,2}=p^2(1+\alpha\frac{E\{\eta G(\eta)\}}{\sigma_\eta^{\,2}})=p^2(1+3\alpha\sigma^2_\eta).\,, \quad (28)$$

where σ_η is the standard deviation of $\eta(t)$. This equation shows how the nonlinear component of the stiffness element affects the value of p_e.

2. NUMERICAL RESULTS

Consider in this example

$$m=1kg, k=36\frac{N}{m}, c=4\frac{Ns}{m}, \alpha=3m^{-2}. \quad (29)$$

Let us set the subsequent values of excitation parameters

$$D = 50N^2, \lambda = 1s^{-1}, \beta = 3s^{-1}. \tag{30}$$

Obtain:

$$\sigma^2_\eta = \frac{214 + 1620\sigma^2_\eta}{2692 \cdot 10^3 \sigma^6_\eta + 7481{,}91 \cdot 10^4 \sigma^4_\eta + 8354{,}52 \cdot 10^3 \sigma^2_\eta + 3240}, \tag{31}$$

or

$$2692 \cdot 10^3 \sigma^8_\eta + 7481{,}91 \cdot 10^4 \sigma^6_\eta + 8354{,}52 \cdot 10^3 \sigma^4_\eta + 1620\sigma^2_\eta - 214 = 0, \tag{32}$$

$$\sigma^2_\eta = 0{,}052m^2. \tag{33}$$

Substituting the equation (33) into equation (28), obtain

$$p_e^{\,2} = p^2(1 + 3\alpha\sigma^2_\eta) = 7{,}26s^{-1}. \tag{34}$$

In literature, very little attention has been paid to the frequency domain characteristics of nonlinear, dynamic systems excited by stochastic processes. It will be shown that this information can be of great value for the understanding of the system's stochastic behaviour.

In the figures 1, 2, 3, 4 and 5, the power spectral density of the excitation, $S_F\,[N^2 \cdot s]$, is plotted for the different parameters D, λ, β. Figure 6 describes the harmonic peak with the same parameter values.

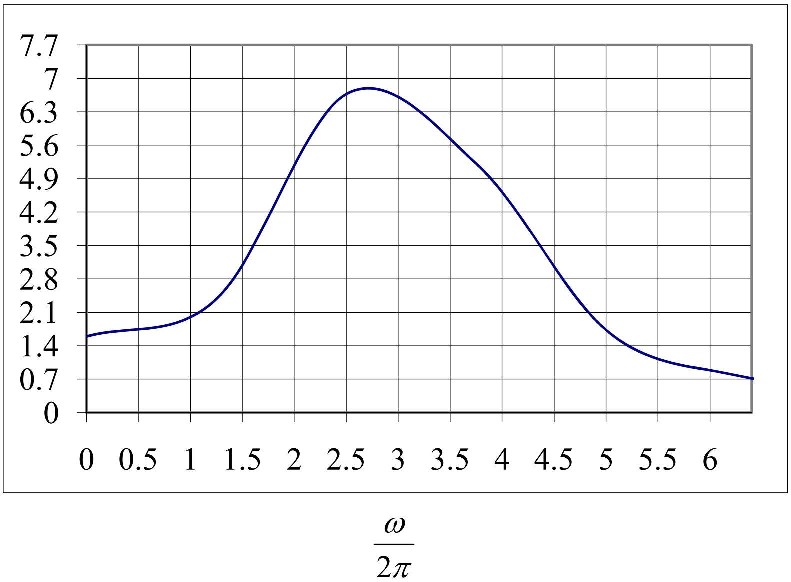
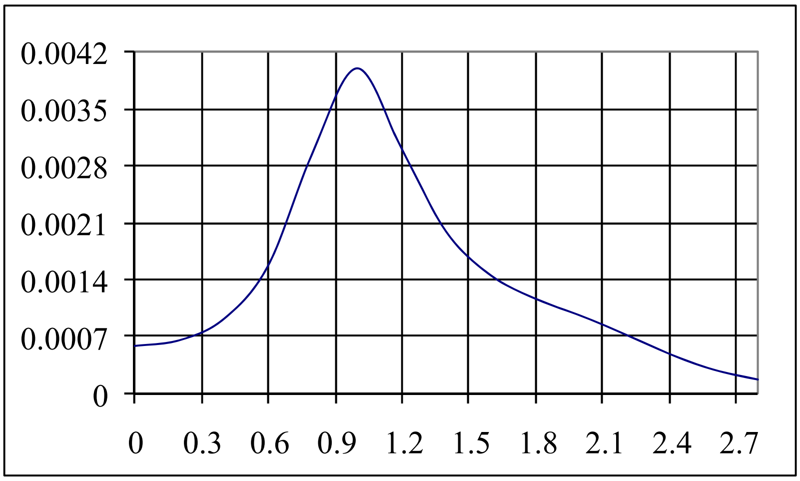


Fig. 3. The power spectral density $S_F\,[N^2 \cdot s]$ of excitation for

$D = 50N^2, \lambda = 1s^{-1}, \beta = 3s^{-1},\ m = 1kg, k = 36\frac{N}{m}, c = 4\frac{Ns}{m}, \alpha = 3m^{-2}.$



$$\frac{\omega}{2\pi}$$

Fig.4. The power spectral density $S_\eta\ [m^2 \cdot s]$ of response for

$$m = 1kg, k = 36\frac{N}{m}, c = 4\frac{Ns}{m}, \alpha = 3m^{-2}.$$

3. CONCLUSIONS

The statistical linearization technique can also tackle a wide variety of problems and also provides approximate information on the frequency domain characteristics of the stochastic response. In this technique, a linear model, which optimally is the original, nonlinear system (in some statistical sense), is constructed. Due to the fact that response statistics of such a model can, in general, be evaluated analytically, statistical linearization is computationally very efficient. However, it only provides accurate approximation of the response statistics for weakly nonlinear systems. In this chapter, it is shown that the statistical linearization technique structurally underestimates the variance of the response of the piece-wise linear system (even for a moderate nonlinearity). This is dangerous when these estimates are used in failure criteria for practical systems. The cause for this underestimation of the variance can be found by comparing accurate, simulated frequency domain characteristics with those determined using the linear model.

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