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APPROXIMATE SOLUTION OF THE NON-LINEAR EQUATION FOR A SYSTEM UNDER A RANDOM VIBRATION

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ABSTRACT - In structural and mechanical engineering, problems involving unpredictable or stochastic variables or processes are frequently encountered, and in these cases a probabilistic analysis may be most rational way of approaching the problem. In many problems involving the dynamical behaviour of mechanical systems, the dominating source of uncertainty or unpredictability is the excitation. If the excitation is given in terms of a stochastic process, the response of the mechanical system is also a stochastic process. In order to assess the probability of accurance of extreme events and evaluate possible fatigue damage in the structure, it is necessary to be able to evaluate the response statistics with reasonable accuracy. In this article the stationary density of the response and the power spectral density of the response is addressed.

1. SYSTEM MODEL

A system with a relatively simple non-linear behaviour is an oscillator with linear stiffness and power law viscous damping. If an oscillator of this type is excited by additive white noise the equation of motion can be expressed as

$$\ddot{x}(t)+h\left(\dot{x}(t)\right)\dot{x}(t)+p^2x(t)=w\left(t\right), \tag{1}$$

where $x(t)$ is the displacement and a lot indicates the derivative with respect to time. The set of conditions that guarantees the existence of the Fourier transform are the Dirichlet conditions, which may be expressed as: the signal $x(t)$ has a finite number of finite discontinuities, the signal $x(t)$ contains a finite number of maxima and minima and it is absolutely integrable, that is

$$\int_{-\infty}^{\infty}\left|x(t)\right|dt<\infty . \tag{2}$$

The second term on the left hand side of of the equation represents the forces due to damping and $h\left(\dot{x}(t)\right)$ is termed the damping function. The right hand side of the equation is the external white noise excitation for the intensity S_0

$$h(\dot{x})=c\,\dot{x}^{\lambda_1-1} . \tag{3}$$

The equation process is then multiplied by $\frac{t_0^2}{x_0}$, whereby the following non-dimensional equation of motion is obtained

$$\ddot{\eta}+f(\dot{\eta})\dot{\eta}+\eta=U(\tau), \tag{4}$$

where $U(\tau)$ is a unit white noise with intensity $\frac{1}{2\pi}$. The non-dimensional time τ, displacement η and velocity $\dot{\eta}$ are given by

$$\tau=\frac{t}{t_0}, \eta=\frac{x}{x_0}, \dot{\eta}=\frac{d\eta}{d\tau}. \tag{5}$$

A dot used in connection with η this indicates the derivative with respect to τ. The non-dimensional damping function is given by

$$f(\dot{\eta})\dot{\eta}=\lambda_2\dot{\eta}^{\lambda_1}, \tag{6}$$

where λ_2 is a non-dimensional damping coefficient

$$\lambda_2=\frac{c}{p}\left(\frac{2\pi S_0}{p}\right)^{\frac{\lambda_1-1}{2}}. \tag{7}$$

The equation of motion is seen only to depend on the parameters λ_1 and λ_2 in this form. In the method of equivalent non-linearization, Caughey [1], an equivalent non-linear system is introduced as

$$\ddot{\eta}(\tau)+f_e(E_m)\dot{\eta}(\tau)+\eta(\tau)=U(\tau),\ E_m=\frac{1}{2}\dot{\eta}^2+\frac{1}{2}\eta^2, \tag{8}$$

where E_m is the mechanical energy non-dimensional. $f_e(E_m)$ is a non-dimensional equivalent damping function, which is assumed to be function of the mechanical energy only. The difference between the nonlinear stiffness and linear stiffness terms is

$$\varepsilon=f(\dot{\eta})\dot{\eta}-f_e(E_m)\dot{\eta}. \tag{9}$$

We obtain

$$E\left[\varepsilon^2|E_m\right]=E\left[\dot{\eta}^2|E_m\right]f_e^2-2E\left[f\dot{\eta}^2|E_m\right]f_e+E\left[f^2\dot{\eta}^2|E_m\right]. \tag{10}$$

The value of $f_e(E_m)$ can be obtained by minimizing the expectation of the square error

$$E\left[\varepsilon^2|E_m\right]=0. \tag{11}$$

The equivalent damping function is evaluated by

$$f_e\left(E_m\right)=\frac{E\left[f\left(\dot{\eta}\right)\dot{\eta}^2 \mid E_m\right]}{E\left[\dot{\eta}^2 \mid E_m\right]}, \tag{12}$$

where $E[\ ]$ is the mean value for a given level.

The equivalent damping function [2,3] is now evaluated considering the harmonic motion given by

$$f_e\left(E_m\right)=\frac{\frac{1}{2\pi}\int_0^{2\pi}\lambda_2\dot{\eta}^{\lambda_1+1}d\tau}{\frac{1}{2\pi}\int_0^{2\pi}\dot{\eta}^2 d\tau}=aE_m^{\frac{\lambda_1-1}{2}}, \tag{13}$$

where

$$a=\frac{\lambda_2 2^{\frac{\lambda_1+1}{2}}}{\sqrt{\pi}}\frac{\Gamma\left(\frac{\lambda_1}{2}+1\right)}{\Gamma\left(\frac{\lambda_1}{2}+\frac{3}{2}\right)} \tag{14}$$

and $\Gamma(.)$ is Gamma function.

Using by

$$\eta=\sqrt{2E_m}\sin\tau, \tag{15}$$

the equation of motion (8) is

$$-\sqrt{2E_m}\sin\tau+f_e\left(E_m\right)\sqrt{2E_m}\cos\tau+\sqrt{2E_m}\sin\tau=U\left(\tau\right). \tag{16}$$

The equation Fokker-Planck-Kolmogorov [2,4] are

$$\frac{d}{dE_m}\left(f_e\left(E_m\right)\dot{\eta}P_{E_m}+\pi S_0\dot{\eta}\frac{dP_{E_m}}{dE_m}\right)=0, \tag{17}$$

where P_{E_m} is the probability density of the system.

Obtain for the density function

$$P_{E_m}\left(E_m\right)=Ce^{-\frac{\int_o^{E_m} f_e(u)du}{\pi S_0}}, \tag{18}$$

where C is a constant which normalises the density function.

Substituting the equation (13) and (14.) into equation (18) obtain

$$P_{E_m}\left(E_m\right)=Ce^{-\frac{2\lambda_2 2^{\frac{\lambda_1+1}{2}}\Gamma\left(\frac{\lambda_1}{2}+1\right)}{\pi\sqrt{\pi}(\lambda_1+1)S_0\Gamma\left(\frac{\lambda_1}{2}+\frac{3}{2}\right)}E_m^{\frac{\lambda_1+1}{2}}}. \tag{19}$$

The power spectral density of response is [2,5]

$$S_\eta\left(r\right)=\int_0^\infty S_\eta\left(r|E_m\right)P_{E_m}\left(E_m\right)dE_m, \tag{20}$$

where

$$S_\eta\left(r|E_m\right)=\frac{f_e\left(E_m\right)E_m}{\pi}\frac{1}{\left(1-r^2\right)^2+f_e^2\left(E_m\right)r^2}, r=\frac{\omega}{p}. \tag{21}$$

Substituting the equation (19) and (21) into equation (20), we obtain for the power spectral density of response

$$S_\eta\left(r\right)=\frac{C}{\pi}\int_0^\infty\frac{f_e\left(E_m\right)E_m e^{-\frac{2\lambda_2 2^{\frac{\lambda_1+1}{2}}\Gamma\left(\frac{\lambda_1}{2}+1\right)}{\pi\sqrt{\pi}S_0(\lambda_1+1)\Gamma\left(\frac{\lambda_1}{2}+\frac{3}{2}\right)}E_m^{\frac{\lambda_1+1}{2}}}}{\left(1-r^2\right)^2+f_e^2\left(E_m\right)r^2}dE_m. \tag{22}$$

Following the equation (19) and (22), we have

$$S_\eta\left(r\right)=\frac{C\lambda_2 2^{\frac{\lambda_1+1}{2}}}{\pi\sqrt{\pi}}\frac{\Gamma\left(\frac{\lambda_1}{2}+1\right)}{\Gamma\left(\frac{\lambda_1}{2}+\frac{3}{2}\right)}\int_0^\infty\frac{E_m^{\frac{\lambda_1+1}{2}}e^{-\frac{2\lambda_2 2^{\frac{\lambda_1+1}{2}}\Gamma\left(\frac{\lambda_1}{2}+1\right)}{\pi\sqrt{\pi}S_0(\lambda_1+1)\Gamma\left(\frac{\lambda_1}{2}+\frac{3}{2}\right)}E_m^{\frac{\lambda_1+1}{2}}}}{\left(1-r^2\right)^2+a^2E_m^{\lambda_1-1}r^2}dE_m. \tag{23}$$

2. EXAMPLE: THE RANDOM DUFFING OSCILLATOR

For convenience, the Duffing oscillator has been used to illustrate this procedure here. Consider the Duffing equation of motion

$$\ddot{\eta}+\lambda_2\dot{\eta}^3+\eta=U\left(\tau\right), \tag{24}$$

with parameter $m=1000kg$, $k=36000\frac{N}{m}$, $c=2400\frac{Ns}{m}$, $\lambda_1=3$.

Obtain $\lambda_2=\frac{c}{p}\left(\frac{2\pi S_0}{p}\right)^{\frac{\lambda_1-1}{2}}$ =0,066 and for the undamped natural frequency

$$p=\sqrt{\frac{k}{m}}=6s^{-1}. \tag{25}$$

The equivalent linear systems with random coefficients [4] is written as

$$\ddot{\eta}(\tau)+aE_m\,\dot{\eta}(\tau)+\eta(\tau)=U(\tau), \tag{26}$$

where $E_m=\frac{1}{2}\dot{\eta}^2+\frac{1}{2}\eta^2$ is the mechanical energy non-dimensional and

$$a=\frac{\lambda_2 2^{\frac{\lambda_1+1}{2}}}{\sqrt{\pi}}\frac{\Gamma\left(\frac{\lambda_1}{2}+1\right)}{\Gamma\left(\frac{\lambda_1}{2}+\frac{3}{2}\right)}=0,066\frac{\frac{\lambda_1}{2}\Gamma\left(\frac{\lambda_1}{2}\right)}{\left(\frac{\lambda_1}{2}+\frac{1}{2}\right)\Gamma\left(\frac{\lambda_1}{2}+\frac{1}{2}\right)}=0,092. \tag{27}$$

The forward Fokker Planck equation [6, 9] which governs the transitional probability density function P_{E_m} of system (4) is obtained as follows

$$\frac{d}{dE}\left(aE\dot{\eta}P_{E_m}+\pi S_0\dot{\eta}\frac{dP_{E_m}}{dE_m}\right)=0. \tag{28}$$

We obtain a solution for the stationary joint probability density function P_{E_m} as

$$P_{E_m}\left(E_m\right)=Ce^{-\frac{\int_o^{E_m} aE_m dE_m}{\pi S_0}}, \tag{29}$$

where C is normalisation constants.
We obtain

$$P_{E_m}\left(E_m\right)=Ce^{-\frac{aE_m{}^2}{2\pi S_0}}. \tag{30}$$

We obtain a solution for the stationary joint probability density function P_{E_m} as

$$P_{E_m}\left(E_m\right)=\frac{1}{\pi}\sqrt{\frac{a}{2S_o}}e^{-\frac{aE_m{}^2}{2\pi S_0}}, \tag{31}$$

where the constant of normalises is

$$C=\frac{1}{\pi}\sqrt{\frac{a}{2S_o}}. \tag{32}$$

For intensity white noise $\frac{1}{2\pi}$, we obtain

$$P_{E_m}\left(E_m\right)=0,171e^{-0,092E_m{}^2}. \tag{33}$$

The power spectral density of response is

$$S_\eta\left(r\right)=\int_0^\infty S_\eta\left(r|E_m\right)P_{E_m}\left(E_m\right)dE_m, \tag{34}$$

where

$$S_\eta\left(r|E_m\right)=\frac{aE_m{}^2}{\pi}\frac{1}{\left(1-r^2\right)^2+a^2E_m{}^2r^2}. \tag{35}$$

We obtain for the power spectral density of response [4,5]

$$S_\eta\left(r\right)=\frac{a}{\pi^2}\int_0^\infty \frac{E_m{}^2}{\left(1-r^2\right)^2+a^2E_m{}^2r^2}\sqrt{\frac{a}{2S_o}}e^{-\frac{aE_m{}^2}{2\pi S_0}}dE_m. \tag{36}$$

3. THE NUMERICAL RESULTS

For $m=1000kg$ and white noise intensity $\frac{1}{2\pi}\ \frac{N\cdot m}{s}$, $S_F=159,2N^2\cdot s$, we obtain

$$S_\eta\left(r\right)=9,3\cdot 10^{-6}\int_0^\infty \frac{E_m{}^2}{\left(1-r^2\right)^2+0,0084E_m{}^2r^2}e^{-0,092E_m{}^2}dE_m. \qquad [m^2\cdot s] \quad .(37)$$

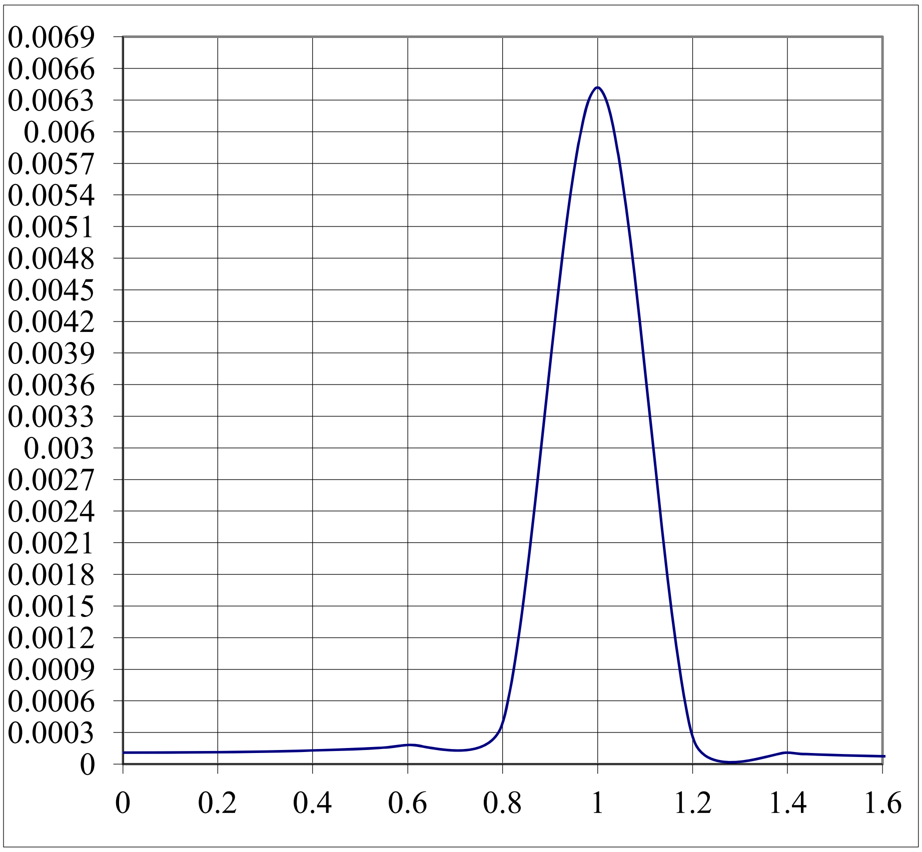

Fig. 1- The power of spectral density $S_\eta[m^2 \cdot s]$

We know this

$$\int_0^\infty \frac{E_m{}^2}{uE_m{}^2+s} e^{-qE_m{}^2} dE_m = \frac{1}{s}\left(1+s-\frac{u}{qs}\right)\sqrt{\frac{\pi}{q}}\,, \tag{38}$$

where

$$q=0,092; \quad u=0,0084r^2; \quad s=(1-r^2)^2\,. \tag{39}$$

4. CONCLUSION

This method is restricted to wideband excitations and lightly damped systems, such that the response can be taken to be a narrow band process. No general method is available at present to obtain the response probability density function and the power spectral density of a non-linear system under a given arbitrary Gaussian random input. Detailed numerical results are presented for of nonlinear oscillators under white noise excitation. The exact probability structure of this special input is found. However, the condition derived is only sufficient, and hence the solution presented is not unique.

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